Expert Systems with Applications 42 (2015) 6522–6529

Contents lists available at ScienceDirect

Expert Systems with Applications

journal homepage: www.elsevier.com/locate/eswa

The slack-based measure model based on supporting hyperplanes of production possibility set

A. Hadi-Vencheh [a,*], J. Jablonsky [b], A. Esmaeilzadeh [a]

[a] Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran
[b] University of Economics, Department of Econometrics, W. Churchill Sq. 4, 130 67 Praha, Czech Republic

ARTICLE INFO

Article history:
Available online 17 April 2015

Keywords:
Data Envelopment Analysis (DEA)
Slack based measure (SBM) model
Hyperplane
Production possibility set (PPS)

ABSTRACT

The original slack-based measure (SBM) model evaluates the efficiency of decision making units (DMUs) referring to the furthest frontier point within a range. Hence the projection may go to a remote point on the efficient frontier which may be inappropriate as a reference point. In this paper we propose a new variant for the improvement of efficiency scores in SBM models. It is based on the determination of strong hyperplanes of the production possibility set (PPS). The approach presented here improves the currently used Tone method both, from the time consumption and computational points of view. Comparative examples, as well as a case study, are given to illustrate the new procedure.

1. Introduction

Data Envelopment Analysis (DEA) is a relatively new "data oriented" approach for the evaluation of the performance of a set of entities called Decision Making Units (DMUs), which transform multiple inputs into multiple outputs. The DEA research started by publication of the essential paper (Charnes, Cooper, & Rhodes, 1978). The definition of DMUs is very generic and flexible. Recent years have seen a great variety of applications of DEA models for performance and efficiency evaluation of many different kinds of entities engaged in many different activities and contexts, in many different countries (Akçay, Ertek, & Büyüközkan, 2012;Bayraktar, Tatoglu, Turkyilmaz, Delen, & Zaim, 2012;Charles & Zegarra, 2014;Cooper, Seiford, & Tone, 1999;Emrouznejad, Parker, & Tavares, 2008; Rezaei, Ortt, & Scholten, 2012;Wanke & Barros, 2014).

As it is known in DEA, the observed DMUs define the production possibility set (PPS). The PPS is a polyhedral convex set whose vertices correspond to the efficient DMUs. The DEA models find the projection of the inefficient DMUs on the efficient frontier of the PPS. That is why the hyperplanes of the PPS are helpful and important in the process of efficiency evaluation of DMUs, as well as in sensitivity and stability analysis (Jahanshahloo, Hosseinzadeh Lotfi, Shoja, Sanei, & Tohidi, 2005a;Khanjani Shiraz, Charles, & Jalalzadeh, 2014). Unfortunately, there are only a few papers in existence which deal with hyperplanes of the PPS and their usage. For instance, Jahanshahloo, Hosseinzadeh Lotfi, and Zohrehbandian (2005b) proposed a method for obtaining the efficient frontier using the integer programming model with binary variables. Yu, Wei, Brockett, and Zhou (1996) studied the structural properties of DEA efficient surfaces of the PPS under the generalized DEA model. Jahanshahloo, Hosseinzadeh Lotfi, Zhiani Rezai, and Rezai Balf (2007), Jahanshahloo, Shirzadi, and Mirdehghan (2009) and Jahanshahloo, Hosseinzadeh Lotfi, and Akbarian (2010) developed the algorithms which are used to find defining hyperplanes of the PPS. Amatatsu and Ueda (2012) show an alternative use of the efficient facets in DEA. Specifically, they indicate that once all facets of the DEA technology is identified, decision maker is able to estimate the potential changes in some inputs and outputs, while fixing other inputs and outputs. Aparicio and Pastor (2014) show least distance measures based on Hölder norms satisfy neither weak nor strong monotonicity on the strongly efficient frontier. They study Hölder distance functions and show why strong monotonicity fails. Along this line, they provide a solution for output-oriented models that allows assuring strong monotonicity on the strongly efficient frontier. Aghayi and Gheleyj Beigi (2014) show that the strong (weak) defining hyperplane is supporting and there exists, at least, one affine independent set with $m+s$ elements of extreme efficient DMUs (extreme efficient and weak efficient virtual DMUs) where m and s are the number of inputs and outputs, respectively. Nasrabadi, Dehnokhalaj, and Soleimani-damaneh (2014) characterize a subset of the production possibility set consisting of production points whose radial projection points lie on the same supporting hyperplane of the PPS. To this end, they consider the CCR and BCC models and establish some theoretical results by utilizing linear programming-based techniques. Determining such a subset of the PPS provides a means to perform

* Corresponding author.
E-mail addresses: ahadi@khuisf.ac.ir (A. Hadi-Vencheh), Jablon@vse.cz (J. Jablonsky), esesmailzadeh1386@gmail.com (A. Esmaeilzadeh).

http://dx.doi.org/10.1016/j.eswa.2015.03.032
0957-4174/

sensitivity analysis of inefficient units. Aparicio and Pastor (2013) show that the Russell output measure of technical efficiency based on closest targets is strongly monotonic when dealing with a full-dimensional strong efficient frontier. This is achieved by replacing non-efficient faces, in the Pareto sense, by linear combinations from the full-dimensional part of the efficient frontier, i.e., extending the efficient facets of the original DEA technology.

One of the main tasks in the DEA models is to find the efficiency score of all (efficient and inefficient) DMUs. Generally speaking, calculation of the efficiency score of the DMUs is investigated in terms of the following aspects. The first aspect is the computational time. It is important to get the results for all DMUs in a short time. The second aspect is computational accuracy; i.e., to calculate the efficiency score of inefficient DMUs with the least possible error. The DEA models cannot satisfy both aspects simultaneously. They just take into account one aspect according to the decision maker's point of view. In most situations, the aim of the decision maker is not just to split the DMUs into inefficient and efficient classes. The aim is, rather, to find ways to improving the efficiency score of inefficient DMUs with the least possible energy, cost or other inputs. It is clear that the calculation of the exact value of the efficiency score of DMUs is very important. The original SBM model evaluates the efficiency of DMUs referring to the furthest frontier point within a given range. Whereas the classical SBM-model projects inefficient DMUs on the efficiency frontier. This may that this projection be a remote point on the frontier. In other words, it is possible that the obtained projection does not located to the closest supporting hyperplane of PPS. This is problematic when aim is removing the inefficiency. That is, when the projected point is remote then removing the inefficiency is difficult.

In an effort to overcome this shortcoming, Tone proposed four variants of the SBM model (Tone, 2010). Tone's method (Tone, 2010) is very interesting, but he deals only with the efficient part (strong hyperplanes) of the PPS, and ignores the inefficient frontiers. On the other hand due to the number and structure of the themes of the SBM model, proposed by Tone, computational time increases considerably. When the efficiency score is computed in all themes for a particular DMU, then the maximum value is selected as the final efficiency score. The second theme consists in finding the facets of the PPS. For this purpose an algorithm was proposed by Tone (for more details, see Tone, 2010). Unfortunately, creating a computer programming code for this algorithm is impossible. That is why the mentioned algorithm can be only used when the number of DMUs is small. In the third theme, the clustering approach was proposed for using the facets' algorithm, but the question is, what type of clustering is better. Besides this, the obtained efficiency scores are local and restricted to the same cluster, and it is not possible to obtain a global efficiency score across the whole PPS. In the fourth theme, a random direction approach is proposed as a way to find facets. Unfortunately, the number of steps in random directions is unknown and this can lead to undesirably long calculations.

In this paper, we first show that the results of Tone's method on the inefficient part of the PPS, are not better than those on the efficient part. Next, we propose a new procedure for finding all facets of the PPS without any clustering or random search. Our procedure is computationally feasible for a large number of DMUs, and as the result, it reduces the massive enumeration of facets.

The rest of this paper is organized as follows. The following section contains the introductory definitions and preliminaries of the paper. In this section, we review and discuss the Tone's method. Main paper's contribution is included in Section 3, where we propose a method for finding strong supporting hyperplanes of the PPS. The proposed procedure, along with three numerical examples, is illustrated in Section 4. Section 5 contains a case study based on a real data set. Its results allow better understanding of the proposed algorithm. Conclusions as the last section summarize the given results, and presents directions for future research in this field.

2. Preliminaries

Suppose we have n DMUs, where all DMUs ($\text{DMU}_j, j=1,\ldots,n$) have m inputs $\mathbf{x}_{ij}, i=1,\ldots,m$ and s outputs $\mathbf{y}_{rj}, r=1,\ldots,s$. We denote the DMU_j by $(\mathbf{x}_j,\mathbf{y}_j), j=1,\ldots,n$, and the input/output data matrices by $\mathbf{X}=(\mathbf{x}_{ij})\in\mathbb{R}^{m\times n}$ and $\mathbf{Y}=(\mathbf{y}_{rj})\in\mathbb{R}^{s\times n}$, respectively, and assume $(\mathbf{X},\mathbf{Y})>(\mathbf{0},\mathbf{0})$. We define the PPS based on constant returns to scale (CRS) as follows:

$$T_c=\left\{(\mathbf{x},\mathbf{y})|\mathbf{x}\geqslant\sum_{j=1}^{n}\lambda_j\mathbf{x}_j,\quad \mathbf{y}\leqslant\sum_{j=1}^{n}\lambda_j\mathbf{y}_j,\quad \lambda_j\geqslant 0,\quad j=1,\ldots,n\right\}$$

and when we deal with variable returns to scale (VRS) the PPS is defined as:

$$T_v=\left\{(\mathbf{x},\mathbf{y})|\mathbf{x}\geqslant\sum_{j=1}^{n}\lambda_j\mathbf{x}_j,\ \mathbf{y}\leqslant\sum_{j=1}^{n}\lambda_j\mathbf{y}_j,\ \sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\geqslant 0,\ j=1,\ldots,n\right\},$$

in which $\lambda\in\mathbb{R}^n$. We introduce non-negative input and output slacks $\mathbf{S}^-\in\mathbb{R}^m$ and $\mathbf{S}^+\in\mathbb{R}^s$ to express $\mathbf{x}=\mathbf{X}\lambda+\mathbf{S}^-$ and $\mathbf{y}=\mathbf{Y}\lambda-\mathbf{S}^+$. The SBM model is defined as (Tone, 2001):

$$\begin{aligned}
\rho_o=\min&\frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}}\\
s.t.&\sum_{j=1}^{n}\lambda_j x_{ij}+s_i^-=x_{io}\quad i=1,\ldots,m\\
&\sum_{j=1}^{n}\lambda_j y_{rj}-s_r^+=y_{ro}\quad r=1,\ldots,s\\
&\lambda_j\geqslant 0\quad j=1,\ldots,n\\
&s_i^-\geqslant 0\quad i=1,\ldots,m\\
&s_r^+\geqslant 0\quad r=1,\ldots,s
\end{aligned}\tag{1}$$

Definition 1 (*Reference set*). Let $(\rho_o^*,\lambda^*,S^{-*},S^{+*})$ be an optimal solution of the model (1). The reference set for DMU $(\mathbf{x}_o,\mathbf{y}_o)$ is defined by $R=\{j|\lambda_j^*>0, j=1,\ldots,n\}$.

In this paper we consider the following models

$$\begin{aligned}
\max\quad & U\mathbf{y}_o\\
s.t.\quad & V\mathbf{x}_o=1\\
& U\mathbf{y}_j-V\mathbf{x}_j\leq 0\qquad j=1,\ldots,n\\
& U\geqslant\mathbf{0},\quad V\geqslant\mathbf{0}
\end{aligned}\tag{2}$$

$$\begin{aligned}
\max\quad & U\mathbf{y}_o+u_o\\
s.t.\quad & V\mathbf{x}_o=1\\
& U\mathbf{y}_j-V\mathbf{x}_j+u_o\leq 0\qquad j=1,\ldots,n\\
& U\geqslant\mathbf{0},\quad V\geqslant\mathbf{0},\quad u_o\,\texttt{free}
\end{aligned}\tag{3}$$

$$\begin{aligned}
\max\quad & U\mathbf{y}_o+u_{o1}-u_{o2}\\
s.t.\quad & V\mathbf{x}_o=1\\
& U\mathbf{y}_j-V\mathbf{x}_j+u_{o1}-u_{o2}\leq 0\qquad j=1,\ldots,n\\
& U\geqslant\mathbf{0},\quad V\geqslant\mathbf{0},\quad u_{o1}\geqslant 0,\quad u_{o2}\geqslant 0
\end{aligned}\tag{4}$$

Definition 2 (*CRS-efficient*). DMU_o is CRS-efficient, if there exists at least one optimal solution (U^*,V^*) for (2), with $(U^*,V^*)>(\mathbf{0},\mathbf{0})$ such that $U^*\ \mathbf{y}_o$=1, otherwise DMU_o is CRS-inefficient.

Definition 3. (VRS-efficient).DMU_o is VRS-efficient, if there exists at least one optimal solution (U^*,V^*,u_o^*) for (3), with $(U^*,V^*)>(\mathbf{0},\mathbf{0})$ such that $U^*\ \mathbf{y}_o+u_o^*$=1, otherwise DMU_o is VRS-inefficient.

2.1. Tone's method

In this section, we review Tone's method (Tone, 2010). Let $(\mathbf{x}_j, \mathbf{y}_j), j = 1, \ldots, K$, be K DMUs in T_c (T_v). A linear combination of these K DMUs with positive coefficients is defined as

$$(\mathbf{x}_o, \mathbf{y}_o) = \left(\sum_{j=1}^{K} w_j \mathbf{x}_j, \sum_{j=1}^{K} w_j \mathbf{y}_j \right), \quad w_j > 0, \quad j = 1, \ldots, K \tag{5}$$

(if $\text{DMU}_j \in T_v$, then add $\sum_{j=1}^{K} w_j = 1$).

Theorem 1. *If $(\mathbf{x}_o, \mathbf{y}_o)$ defined by (5) is CRS-efficient (VRS-efficient), then there exists a supporting hyperplane to the PPS at $(\mathbf{x}_o, \mathbf{y}_o)$ which also supports PPS at $(\mathbf{x}_j, \mathbf{y}_j), j = 1, \ldots, K$.*

Definition 4 (*Facet*). The supporting hyperplane $V^*\mathbf{x} - U^*\mathbf{y} \le 0$ is a facet of the PPS.

Definition 5 (*Friends*). A subset $\{P_{j_1}, \ldots, P_{j_k}\}$ of $\{P_j\} = \{(\mathbf{x}_j, \mathbf{y}_j)\}$, $j = 1, \ldots, K$ is called friends if a linear combination with positive coefficients of $\{P_{j_1}, \ldots, P_{j_k}\}$ is CRS-efficient (VRS-efficient).

Definition 6 (*Maximal friends*). A friends is called maximal, if any addition of P_j (not in the friends) to the friends, is no more friends.

Definition 7 (*Dominated friends*). A friends is dominated by other friends if the set of DMUs is a subset of the other's.

The steps of Tone's method are as follows.

Step 1. (Finding efficient DMUs): Utilize one of the non-radial models (SBM or Additive) to find efficient DMUs.

Step 2. (Enumeration of facets): Enumerate all facets by applying the following algorithm. This algorithm finds the maximal friends of $P_j = (\mathbf{x}_j, \mathbf{y}_j), j = 1 \ldots, K$.

Algorithm:

```
Begin
  For k = 1 to K
    Find maximum friends of P_k
  Next k
  Delete the dominated friends from the set of friends
  Obtain the set of facets from the final set of friends
End
Subroutine find-maximal-friends of P_k
  Exclude P_1, ..., P_{k-1} from the candidates of friends
  Enumerate all friends of P_k
Remove dominated friends from the set of friends
  Exit sub
```

Step 3. (Evaluation of inefficient DMUs): Let $(\mathbf{x}_o, \mathbf{y}_o)$ be an inefficient DMU. Evaluate its efficiency score as follows. For each facet$(h), h = 1, \ldots, H$, solve the following fractional program:

$$\begin{aligned} \rho_o^{(h)} = \max & \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \\ s.t. & \sum_{j \in R(h)} \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, \ldots, m \\ & \sum_{j \in R(h)} \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \ldots, s \\ & \lambda_j \geqslant 0 \quad j \in R(h) \\ & s_i^- \geqslant 0 \quad i = 1, \ldots, m \\ & s_r^+ \geqslant 0 \quad r = 1, \ldots, s \end{aligned} \tag{6}$$

where $R(h)$ is the set of efficient DMUs that span facet(h). So, the efficiency score of DMU_o is as $\rho_o^{all} = \max_h\{\rho_o^{(h)}\}$.

2.2. Disadvantages of Tone's methods

To reduce computational time and space consumption for large scale problems, Tone proposed two modified versions of his method: (1) Clustering DMUs. (2) Random search. In the first version, using some clustering methods, the DMUs are classified in clusters and the efficiency score of each DMU is computed in the corresponding cluster. In the second version, using the random directions around the efficient DMUs, an approximate method is proposed for finding facets. But each one of these versions has some drawbacks. For example, in version 1, no clustering is appropriate (some clusters do not give the desired solution) and the obtained efficiency score is local, and not global. That means that it is computed based on the same cluster. Because these clusters are selected such a way that: (a) the number of them must be considerable, and: (b) there must be at least one efficient DMU in each cluster. Version 2 is an approximate method, and we do not know the number of iterations for finding approximate directions. Since in all these themes, firstly, we must find the facets of the PPS, and then using the model (6), the efficiency score of DMU_o must be computed based on its effective facet. Note, that one has to solve a model with $m + s + 1$ constraints, and $\|R(h)\| + m + s + 1$ variables. Clearly, for large scale problems this is impossible. Besides, Tone deals only with the efficient part of the frontier, and does not consider the inefficient part. Now, suppose we have only one efficient DMU, then the whole frontier of the PPS is inefficient. What do we do in this situation? How do we deal with the situation where there is only one efficient DMU? Is it possible that the results of Tone's method on the inefficient part of the frontier will be better than those on efficient part? In the next section, we answer these questions. Also, we employ a multiplier form of the models for finding the strong defining hyperplanes of the PPS, which does not need clustering, or random search.

3. The new variation on the theme of the slacks-based measure

In this section we show that it is impossible that the results of Tone's method on the inefficient part of the frontier can be better than those on the efficient part. Then we will propose our own method.

Generally speaking, the supporting hyperplanes of the PPS are divided into two classes. These are: the strong and weak supporting hyperplanes. We show that to improve the SBM model it is enough to consider strong supporting hyperplanes. First we must prove the following lemma.

Lemma 1. *Consider the following linear programming problem*

$$\begin{aligned} & \max \delta = \mathbf{c}\mathbf{x} \\ s.t. \quad & A\mathbf{x} \leqslant \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{aligned} \tag{7}$$

in which $\mathbf{c}$ is a non-negative vector. Let δ^* be the optimal value of this problem and suppose that a variable, say x_k, is deleted from the decision variables of the problem and δ^*_{new} be the optimal value of the new problem. Then, $\delta^* \geqslant \delta^*_{new}$.

Proof:. Let

$$S = \left\{ (x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) : \sum_{j=1}^{n} \mathbf{a}_j x_j \leqslant \mathbf{b}, x_j \geqslant 0 \right\}$$

$$S_{new} = \left\{ (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) : \sum_{\substack{j=1 \\ j \neq k}}^{n} \mathbf{a}_j x_j \leqslant \mathbf{b}, x_j \geqslant 0 \right\}$$

$$SS_{new} = \left\{ (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) : \sum_{j=1}^{n} \mathbf{a}_j x_j \leqslant \mathbf{b}, x_j \geqslant 0 \right\}$$

where $\mathbf{a}_j$ is the jth column of A. In fact S is the feasible region before deleting x_k, S_{new} is the feasible region after deleting x_k and SS_{new} is the projection of S on surface $x_k = 0$. Obviously, $SS_{new} \subseteq S$. Hence $S := SS_{new} \cup SS'_{new}, SS_{new} \cap SS'_{new} = \phi$, where SS'_{new} is the complement of SS_{new}, and let $(\mathbf{x}^*, \delta^*)$ and $(\mathbf{x}^*_{new}, \delta^*_{new})$ be the optimal solution of the old, and the new problem, respectively. Therefore we have two cases: □

(1) $\mathbf{x}^* \in SS_{new} \Rightarrow \delta^*_{new} = \delta^*$.
(2) $\mathbf{x}^* \notin SS_{new} \Rightarrow \mathbf{x}^* \in SS'_{new} \Rightarrow \delta^*_{new} < \delta^*$.

Tone (2010) deals only with the efficient part of the PPS and he ignores the inefficient part. The following theorem shows that the optimal value of the objective function of model (6) on the inefficient part of the PPS is not better than its value on the strong part of the PPS.

Theorem 2. *The optimal value of the objective function of the model (6) on the inefficient part of the PPS is not better than its value on the strong part of the PPS.*

Proof: Let $\overline{H}$ be an inefficient part of the PPS. It is clear that $\overline{H}$ passes through at least one efficient DMU. Without loss of generality, let A be this DMU. There are strong hyperplanes that are adjacent to $\overline{H}$. These strong hyperplanes pass through A and some other efficient DMUs. If we denote the PPS of the model (6) on the inefficient part of the frontier by S', and on the its adjacent efficient hyperplane with S, then $S' \subseteq S$. Now Lemma 1 completes the proof. □

Remark:. According to the above theorem, we conclude that it is enough to obtain strong hyperplanes of the PPS. But, if there exists only one efficient DMU, then we do not have any strong hyperplane. In this case we just use this efficient DMU, instead of $R(h)$, and run the model (6). In Section 5 we illustrate this using a numerical example.

3.1. The proposed method

Now we are ready to present our proposed procedure. Note that according to the above discussion, we just consider strong hyperplanes. The subsequent theorems are taken from Jahanshahloo et al. (2009).

Definition 8 (*Affine independent*). A collection of vectors $a_1, \ldots, a_{k+1}$;of dimension n is called affine independent if $\{a_2 - a_1, \ldots, a_{k+1} - a_1\}$ is linear independent.

Definition 9. H is a strong defining hyperplane of the PPS if it is supporting and there exists at least one affine independent set with $m + s$ elements of strongly efficient DMUs that lie on H.

In the evaluation of $\text{DMU}_j, j \in \{1, \ldots, n\}$, if (U^*, V^*, u_o^*) is an optimal solution of model (3) then $U^*\mathbf{y} - V^*\mathbf{x} + u_o^* = 0$ is a supporting hyperplane on the PPS (Cooper et al., 1999). In Fig. 1, using model (3), it can be seen that there are alternative optimal solutions which define an infinite number of hyperplanes passing through

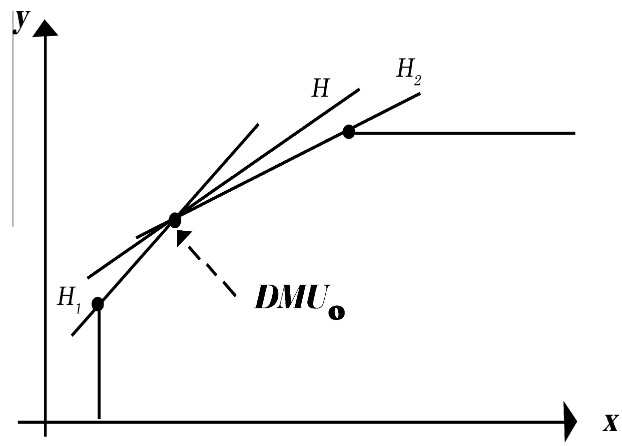

Fig. 1. $H_1 and H_2$ are defining. H is not defining.

DMU_o, of which only two hyperplanes (H_1 *and* H_2) are the defining hyperplanes. Therefore, the number of supporting hyperplanes of the PPS cannot be infinite.

Theorem 3. *Suppose* DMU_k *is strong efficient,* $H_k : U^*\mathbf{y} - V^*\mathbf{x} = 0$ *is a strong defining hyperplane of the PPS of* T_c *if and only if* (U^*, V^*) *is an extreme (basic feasible) optimal solution of model (2) in evaluating* DMU_k *with* $(U^*, V^*) > (\mathbf{0}, \mathbf{0})$.

Theorem 4. *Suppose* DMU_k *is strong efficient,* $H_k : U^*\mathbf{y} - V^*\mathbf{x} + u_o^* = 0$ *is a strong defining hyperplane of the PPS of* T_v *if, and only if,* $(U^*, V^*, u^*_{o1}, u^*_{o2})$ *is an extreme (basic feasible) optimal solution of (4) in evaluating* DMU_k *with* $(U^*, V^*) > (\mathbf{0}, \mathbf{0})$, *where* $u_o^* = u^*_{o1} - u^*_{o2}$ *and* $u^*_{o1} u^*_{o2} = 0$.

Now the steps of the proposed procedure are as follows.

Step 1. Find strong defining hyperplanes of the PPS using the multiplier form of the CCR model.
Using the multiplier form of the CCR model, we find all efficient DMUs. Let DMU_o be a strong efficient DMU, hence in all optimal solutions $(U^*, V^*) > (\mathbf{0}, \mathbf{0})$. Therefore, the binding strong defining hyperplane is like $h : U^*\mathbf{y} - V^*\mathbf{x} = 0$. Then, we examine which other efficient DMUs are active in h. That is, h passes through them. All such DMUs are co-plane and make facet h. We repeat this step for all efficient DMUs to find all strong defining hyperplanes.

Step 2. Find the efficiency score of the inefficient DMUs.
Let DMU_o be inefficient. We find its distance from all strong defining hyperplanes. Consider the strong defining hyperplane which has the least distance to DMU_o and solve the model (6) on this hyperplane. The obtained value of the objective function is the efficiency score of DMU_o.

Remark:. The calculations related to finding an effective, strong defining hyperplane in efficiency of DMU_o is as follows:

$$d_h = \frac{|\overrightarrow{P_0 DMU_o} \cdot \vec{n}_h|}{|\vec{n}_h|}$$

$$d_{h^*} = \min_h d_h$$

in which P_0 is an arbitrary point of the strong defining hyperplane h, $|\overrightarrow{P_0 DMU_o}|$ is a vector from DMU_o to P_0, $\vec{n}_h = (U, -V)$ is a normal vector of the strong defining hyperplane, $|\vec{n}_h| = \sqrt{\sum_i (-v_i)^2 + \sum_j u_j^2}$ is the norm of $\vec{n}_h$, "." shows inner product and h^* is the effective strong defining hyperplane in efficiency of DMU_o. For more information about the least distance, see Briec (1998), Coelli (1998), Briec and Lemaire (1999a), Briec and Lesourd (1999b), Frei and Harker (1999), Cherchye and Van Puyenbroeck (2001), Portela, Borges, and Thanassoulis (2003), Baek and Lee (2009), Amirteimoori and Kordrostami (2010) and Ando, Kai, Maeda, and Sekitani (2012).

4. Numerical examples

In this section, we examine three numerical examples using the proposed model. The examples were taken from Cooper et al. (1999). Through this section and the next section we compute the coefficients of the hyperplanes using the Maple software and fractions are used because if we use decimals then due to round-off errors, some coefficients are lost (became zero).

Example 1. This example shows a situation where all surfaces of the PPS are inefficient.

Consider Table 1. Here we have 8 DMUs, A to H, with one input x and one output y. In this example only one DMU, B, is efficient (Pareto efficient). Here, we do not have a strong hyperplane but there is one strong efficient DMU. Thus, we employ this DMU for our calculations without finding the hyperplane.

Table 2 displays the results. As we see in this table, the proposed model has improved the efficiency score of each inefficient DMU.

Example 2. This example shows the situation in which the frontier of the PPS includes both the efficient and inefficient parts.

Consider Table 3. This table contains 8 DMUs, A to H, with two inputs (x_1, x_2) and one output y. The results are shown in the Table 4. In this example, F and G are weakly efficient. Note, that according to Theorem 2, the efficiency score of these two DMUs has not been improved in the new procedure. Here, the strong defining hyperplanes are $ED : y - \frac{2499999}{5000000}x_1 - \frac{1249999}{1250000}x_2 = 0$ and $CD : y - \frac{1}{12}x_1 - \frac{1}{3}x_2 = 0$.

For example, to compute the efficiency score of H, we calculated its distance to all strong defining hyperplanes and obtained $d_{CD} = 12.05, d_{ED} = 20.15$. Hence, $d_{CD} = 12.05$ is the least distance. Thus CD is the effective strong defining hyperplane in the efficiency score of H, and we implemented the model (6) on this hyperplane for H, and found that $R(CD) = \{C, D\}$ and $\rho^*_H = 0.8571$.

Example 3. Using the new procedure, we obtained the same results as Tone's method did, but more easily and simpler than if we used Tone's method. Note that we do not need to cluster DMUs, or do random research. This example shows this fact. Consider Table 5.

Table 1
Data of Example 1.

DMU		A	B	C	D	E	F	G	H
Input	x	2	3	3	4	5	5	6	8
Output	y	1	3	2	3	4	2	3	5

Table 2
Efficiency scores of Example 1 by SBM and the new procedure.

DMU	SBM	Reference(s)	Efficiency	New procedure	Reference(s)
A	0.6667	B	inefficient	0.7500	B
B	1.0000	B	efficient	1.0000	B
C	0.8000	B	inefficient	0.8333	B
D	0.8571	B	inefficient	0.8750	B
E	0.8889	B	inefficient	0.9000	B
F	0.5714	B	inefficient	0.7000	B
G	0.6667	B	inefficient	0.7500	B
H	0.7692	B	inefficient	0.8125	B

Table 3
Data of Example 2.

	DMU	A	B	C	D	E	F	G	H
Input	x_1	4	7	8	4	2	10	12	10
	x_2	3	3	1	2	4	1	1	1.5
Output	y	1	1	1	1	1	1	1	1

Table 4
Efficiency scores of Example 2 by the new procedure.

DMU	SBM	Rreference(s)	Efficiency	New procedure	Reference(s)	Effective hyperplane
A	0.8333	D	inefficient	0.9231	D, E	ED
B	0.6190	D	inefficient	0.7742	C, D	CD
C	1.0000	C	efficient	1.0000	C	–
D	1.0000	D	efficient	1.0000	D	–
E	1.0000	E	efficient	1.0000	E	–
F	0.9000	C	inefficient	0.9000	C	CD
G	0.8333	C	inefficient	0.8333	C	CD
H	0.7333	C	inefficient	0.8571	C, D	CD

Table 5
Data of Example 3.

	DMU	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
Input	x_1	20	19	25	27	22	55	33	31	30	50	53	38
	x_2	151	131	160	168	158	255	235	206	244	268	306	273
Output	y_1	100	150	160	180	94	230	220	152	190	250	260	250
	y_2	90	50	55	72	66	90	88	80	100	100	147	133

Table 6
Efficiency scores of Example 3 by SBM and new procedure.

DMU	SBM	Reference(s)	Efficiency	New procedure	Reference(s)	Effective hyperplane
A	1.0000	A	strong efficient	1.0000	A	–
B	1.0000	B	strong efficient	1.0000	B	–
C	0.8265	B,L	inefficient	0.8751	D	*BDL*
D	1.0000	D	strong efficient	1.0000	D	–
E	0.7277	B,L	inefficient	0.7682	A	*ADL*
F	0.6857	A,L	inefficient	0.7265	D	*ADL*
G	0.8765	B,L	inefficient	0.9369	D	*BDL*
H	0.7714	L	inefficient	0.8092	D	*BDL*
I	0.9016	A,L	inefficient	0.9212	A,D,L	*ADL*
J	0.7653	B,L	inefficient	0.8103	D	*BDL*
K	0.8619	B,L	inefficient	0.8889	A,D	*ADL*
L	1.0000	L	strong efficient	1.0000	L	–

Table 7
Inputs and outputs for 25 bank branches.

	x_1	x_2	x_3	x_4	y_1	y_2	y_3
Min	3050	3145	3222	3123	5468	4999	4521
MAX	7825	4988	4632	5000	10000	9821	9941
Mean	4182.6	4071.92	3960.56	4055.28	7602.28	7130.84	7383.08
Median	3980	4012	3989	3999	7415	7312	7415
St.dev	927.1577	561.7359	404.4597	614.7520	1447.706	1404.374	1570.330

According to this table we have 12 DMUs. Each one uses two inputs to produce two outputs. Table 6 shows the results obtained. As we see, A, B, D and L are efficient DMUs, while the other DMUs are inefficient. Note, that in the new procedure, the reference set for some inefficient DMUs are different from those determined by the SBM model. Besides, the efficiency score of the inefficient DMUs has been increased in the new procedure. In this example the strong defining hyperplanes are $ADL : \frac{1}{300}y_1 + \frac{3}{400}y_2 - \frac{1}{400}x_1 - \frac{31}{5000}x_2 = 0$ and $BDL : \frac{39}{10000}y_1 + \frac{41}{5000}y_2 - \frac{61}{10000}x_1 - \frac{34}{5000}x_2 = 0$.

Here we take F into consideration. To calculate the efficiency score of this DMU first we compute its distance to all strong defining hyperplanes. These distances are $d_{ADL} = 41.15$ and $d_{BDL} = 60.69$. So $d_{ADL} = 41.15$ is the least distance. Hence ADL is the effective strong defining hyperplane in the efficiency score of F and therefore we implement the model (6) in this hyperplane for F. Here $R(ADL) = \{A, D, L\}$ and $\rho_F^* = 0.7265$.

5. Application of the proposed procedure

In this section, a case study is presented in order to clarify the advantage of the approach. The case study is related to the efficiency evaluation of 25 branches of Refah Bank, Iran. There are 4 inputs as suspicious receivables cost, personnel cost, capital cost and branch equipment cost, respectively, and 3 outputs as incomes, deposits and banking facilities, respectively. Data gathered from the branches are shown in Table 7. For convenience, we use x_1 to x_4 and y_1 to y_3 instead of their actual names.

In order to compute the efficiency scores of DMUs, using the proposed model, first we find the efficient DMUs. Here we employed the multiplier form of the CCR model and found that DMUs $A, D, F, K, O, P, Q, R, S$ and V are efficient DMUs. The strong defining hyperplanes, and the co-plane efficient DMUs are as follows:

$RQD : \frac{1}{100000000000}y_1 + \frac{2951268}{63059137391}y_2 + \frac{43750}{67102035}y_3 - \frac{1}{100000000000}x_1 - \frac{1}{100000000000}x_2 - \frac{12}{40275}x_3 - \frac{1}{100000000000}x_4 = 0$

$RPOA : \frac{9}{97180}y_1 + \frac{1}{100000000000}y_2 + \frac{1}{100000000000}y_3 - \frac{1}{100000000000}x_1 - \frac{1}{100000000000}x_2 - \frac{1}{100000000000}x_3 - \frac{192307}{710000000}x_4 = 0$

$ROKD : \frac{31585093079}{440005891319810}y_1 + \frac{1718064779}{44000589131981}y_2 + \frac{1}{100000000000}y_3 - \frac{28902197530}{440005891319810}x_1 - \frac{1}{100000000000}x_2 - \frac{187368132838}{880011782639620}x_3 - \frac{1}{100000000000}x_4 = 0$

$VPOF : \frac{217609584}{20316628645230}y_1 + \frac{133556339}{1693052387102}y_2 + \frac{173119134}{4063325729046}y_3 - \frac{1}{100000000000}x_1 - \frac{1}{100000000000}x_2 - \frac{1}{100000000000}x_3 - \frac{1}{3470}x_4 = 0$

$SD : \frac{66272956}{695315061150}y_1 + \frac{6979833}{347657530575}y_2 + \frac{1}{100000000000}y_3 - \frac{1}{100000000000}x_1 - \frac{4999999}{15725000000}x_2 - \frac{1}{100000000000}x_3 - \frac{1}{100000000000}x_4 = 0$

$VRQKA : \frac{6830164}{144591468445}y_1 + \frac{1}{100000000000}y_2 + \frac{4454715}{57836587378}y_3 - \frac{9999998811}{34220000000000}x_1 - \frac{1}{100000000000}x_2 - \frac{1}{100000000000}x_3 - \frac{1}{100000000000}x_4 = 0$

To compute the efficiency score of inefficient DMUs, based on the proposed approach, first, we obtain the distance from each inefficient DMU to all the above strong defining hyperplanes. Then we find the minimum value in these distances. This minimum value determines which hyperplane is effective in the efficiency score of the DMU, and we employ this hyperplane for solving the model (6), in order to obtain the efficiency score of the DMU. Note that in the proposed model the effective strong defining hyperplane is obtained using simple mathematical calculations and then the model (6) is used just one time. Whereas in Tone's model the model (6) is used for all facets of the PPS and then the maximum value is used as the efficiency score of the DMU. For instance, in order to find the efficiency score of B we find its distance from all the strong supporting hyperplanes.

Table 8
Efficiency scores of 25 bank branches by SBM and new procedure.

DMU	SBM	Reference(s)	Efficiency	New procedure	Reference(s)	Effective hyperplane
A	1.0000	A	strong efficient	1.0000	A	–
B	0.6199	D	inefficient	0.7515	D,Q	*RQD*
C	0.6180	D	inefficient	0.7532	K,Q,V	*VRQKA*
D	1.0000	D	strong efficient	1.0000	D	–
E	0.7334	D,R	inefficient	0.8061	D,Q,R	*RQD*
F	1.0000	F	strong efficient	1.0000	F	–
G	0.9139	D,O,R	inefficient	0.9450	D,O	*ROKD*
H	0.6440	D, F	inefficient	0.7664	F,P,V	*VOPF*
I	0.7538	R,k,O	inefficient	0.8579	A,R	*VRQKA*
J	0.6599	D	inefficient	0.7296	D,Q,R	*RQD*
K	1.0000	K	strong efficient	1.0000	K	–
L	0.6101	D	inefficient	0.7505	Q, R	*RQD*
M	0.6214	D,R	inefficient	0.6783	K,O,R	*ROKD*
N	0.7687	D	inefficient	0.8573	D,Q	*RDQ*
O	1.0000	O	strong efficient	1.0000	O	–
P	1.0000	P	strong efficient	1.0000	P	–
Q	1.0000	Q	strong efficient	1.0000	Q	–
R	1.0000	R	strong efficient	1.0000	R	–
S	1.0000	S	strong efficient	1.0000	s	–
T	0.7509	D,O,R	inefficient	0.8669	K,R	*ROKD*
U	0.8254	F,O,R	inefficient	0.8428	A,P,R	*RPOA*
V	1.0000	V	strong efficient	1.0000	V	–
W	0.9442	D,K,Q,R,V	inefficient	0.9701	V,D, A	*VRQKA*
X	0.5020	D	inefficient	0.5673	D,Q	*RQD*
Y	0.8483	D,F,O,R	inefficient	0.9121	D,R,O	*ROKD*

$d_{VRQKA} = 1254.23$, $d_{SD} = 2832.08$, $d_{ROKD} = 1858.84$ $d_{RPOA} = 2134.88$, $d_{RQD} = 97.99$, $d_{VOPF} = 2373.05$

That is, $d_{RQD} = 97.99$ is the least distance. Hence, the effective strong defining hyperplane in the efficiency score of DMU_B is *RQD*, and we solve the model (6) on this hyperplane, and $R(RQD) = \{R, Q, D\}$ and $\rho_B^* = 0.7515$. We use the same method to find the efficiency scores of the other inefficient DMUs. Table 8 displays the efficiency scores of 25 bank branches. Columns two and three of this Table show the efficiency score of each DMU, calculated by the *SBM* and its reference set, respectively. Column four shows the type of efficiency for each DMU. Columns five, six and seven of Table 8 show the efficiency score obtained by the proposed model, reference set and effective strong defining hyperplanes in the efficiency of each DMU, respectively. As we see, the efficiency scores obtained by the proposed model is greater than those obtained by the SBM model.

6. Conclusion

Decision makers (DMUs) needs some strategies to improve performance of their organization. One of them is determining inefficient DMUs. There are several DEA models for this purpose. Most of DEA models, such as classical SBM model, projects inefficient DMU on the frontier of PPS. It may that this projection may lead to a remote point on the frontier and as a result removing the inefficiency may be fail. Here, we introduced a method for the analysis and improvement of the efficiency scores of DMUs in the SBM model. We also proposed a new procedure for finding all strong defining hyperplanes of the PPS. The proposed method is easier than Tone's method, and does not need any clustering of DMUs or any random search. In comparison with Tone's method, the proposed approach can be easily implemented. This allows the application of the proposed method for large scale problems. The applicability and advantages of the algorithms were illustrated on several numerical examples that were completed by a case study with a real economic background.

The limitation of our method is that in the first stage the multiplier model is employed to find all supporting hyperplanes. However, this strategy increase simplicity of the method, but it needs to run multiplier form of the CCR-model and model (6). Hence the computational time is increased. On the other hance, due to importance of strong supporting hyperplanes it is necessary to use softwares which use fractional coefficients to avoid loss data.

Further research in this field can be focused on the wider applications of the proposed algorithms, as well as comparison of their results with other DEA models. Another research direction could be the extension of the proposed procedures for the analysis and ranking of efficient DMUs. Besides, strong supporting hyperplanes play an important role in efficiency estimation. So far, researchers just use them for efficiency analysis and less attention pay to finding them. Therefore as an another future research field, proposing some new simple methods for finding strong supporting hyperplanes is suggested.

Acknowledgement

The authors are grateful for the constructive comments and suggestions made by the two anonymous referees and the Editor-in- Chief, Prof. Binshan Lin. The research for second author is supported by the Czech Science Foundation, project No. P403/12/1387.

References

Aghayi, N., & Gheleyj Beigi, Z. (2014). Identification of strong and weak hyperplanes in data envelopment analysis. *Procedia-Social and Behavioral Sciences, 129*, 365–371.

Akçay, A. E., Ertek, G., & Büyüközkan, G. (2012). Analyzing the solutions of DEA through information visualization and data mining techniques: Smart DEA framework. *Expert Systems with Applications, 39*(9), 7763–7775.

Amatatsu, H., & Ueda, T. (2012). Measurement of simultaneous scale and mix changes in inputs and outputs using DEA facets and RTS. *European Journal of Operational Research, 223*, 752–761.

Amirteimoori, A., & Kordrostami, S. (2010). A Euclidean distance-based measure of efficiency in data envelopment analysis. *Optimization, 59*, 985–996.

Ando, K., Kai, A., Maeda, Y., & Sekitani, K. (2012). Least distance based inefficiency measures on the Pareto-efficient frontier in DEA. *Journal of the Operations Research Society of Japan, 55*(1), 73–91.

Aparicio, J., & Pastor, J. T. (2014). Closest targets and strong monotonicity on the strongly efficient frontier in DEA. *Omega, 44*, 51–57.

Aparicio, J., & Pastor, J. T. (2013). A well-defined efficiency measure for dealing with closest targets in DEA. *Applied Mathematics and Computation, 219*, 9142–9154.

Baek, C., & Lee, J. (2009). The relevance of DEA benchmarking information and the least-distance measure. *Mathematical and Computer Modelling, 49*, 265–275.

Bayraktar, E., Tatoglu, E., Turkyilmaz, A., Delen, D., & Zaim, S. (2012). Measuring the efficiency of customer satisfaction and loyalty for mobile phone brands with DEA. *Expert Systems with Applications, 39*(1), 99–106.

Briec, W. (1998). Hölder distance functions and measurement of technical efficiency. *Journal of Productivity Analysis, 11*, 111–131.

Briec, W., & Lemaire, B. (1999a). Technical efficiency and distance to a reverse convex set. *European Journal of Operational Research, 114*, 178–187.

Briec, W., & Lesourd, J. B. (1999b). Metric distance function and profit: some duality results. *Journal of Optimization Theory and Applications, 101*, 15–33.

Charles, V., & Zegarra, L. F. (2014). Measuring regional competitiveness through data envelopment analysis: A peruvian case. *Expert Systems with Applications, 41*(11), 5371–5381.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research, 2*(6), 429–444.

Cherchye, L., & Van Puyenbroeck, T. (2001). A comment on multi-stage DEA methodology. *Operations Research Letters, 28*, 93–98.

Coelli, T. (1998). A multi-stage methodology for the solution of orientated DEA models. *Operations Research Letters, 23*, 143–149.

Cooper, W. W., Seiford, L. M., & Tone, K. (1999). *Data envelopment analysis: A software*. Kluwer Academic Publishers..

Emrouznejad, A., Parker, B. R., & Tavares, G. (2008). Evaluation of research in efficiency and productivity: A survey and analysis of the first 30 years of scholarly literature in DEA. *Socio-Economic Planning Sciences, 42*(3), 151–157.

Frei, F. X., & Harker, P. T. (1999). Projections onto efficient frontiers: Theoretical and computational extensions to DEA. *Journal of Productivity Analysis, 11*, 275–300.

Jahanshahloo, G. R., Hosseinzadeh Lotfi, F., & Akbarian, D. (2010). Finding weak defining hyperplanes of PPS of the BCC model. *Applied Mathematical Modelling, 34*, 3321–3332.

Jahanshahloo, G. R., Hosseinzadeh Lotfi, F., Shoja, N., Sanei, M., & Tohidi, G. (2005a). Sensitivity and stability analysis in data envelopment analysis. *Journal of the Operational Research Society, 56*, 342–345.

Jahanshahloo, G. R., Hosseinzadeh Lotfi, F., Zhiani Rezai, H., & Rezai Balf, F. (2007). Finding strong defining hyperplanes of production possibility set. *European Journal of Operational Research, 177*(1), 42–54.

Jahanshahloo, G. R., Hosseinzadeh Lotfi, F., & Zohrehbandian, M. (2005b). Finding the piecewise linear frontier production function in data envelopment analysis. *Applied Mathematics and Computation, 163*(1), 483–488.

Jahanshahloo, G. R., Shirzadi, A., & Mirdehghan, S. M. (2009). Finding strong defining hyperplanes of PPS using multiplier form. *European Journal of Operational Research, 194*, 933–938.

Khanjani Shiraz, R., Charles, V., & Jalalzadeh, L. (2014). Fuzzy rough DEA model: A possibility and expected value approaches. *Expert Systems with Applications, 41*(2), 434–444.

Nasrabadi, N., Dehnokhalaj, A., & Soleimani-damaneh, M. (2014). Characterizing a subset of the PPS maintaining the reference hyperplane of the radial projection point. *Journal of the Operational Research Society, 65*, 1876–1885.

Portela, M. C. A. S., Borges, P. C., & Thanassoulis, E. (2003). Finding closest targets in non-oriented dea models: the case of convex and non-convex technologies. *Journal of Productivity Analysis, 19*, 251–269.

Rezaei, J., Ortt, R., & Scholten, V. (2012). Measuring entrepreneurship: Expert-based vs. data-based methodologies. *Expert Systems with Applications, 39*(4), 4063–4074.

Tone, K. (2001). A slacks-based measure of efficiency in data envelopment analysis. *European Journal of Operational Research, 130*, 498–509.

Tone, K. (2010). Variations on the theme of slacks-based measure of efficiency in DEA. *European Journal of Operational Research, 200*, 901–907.

Wanke, P., & Barros, C. (2014). Two-stage DEA: An application to major Brazilian banks. *Expert Systems with Applications, 41*(5), 2337–2344.

Yu, G., Wei, Q., Brockett, P., & Zhou, L. (1996). Construction of all DEA efficient surfaces of the production possibility set under the generalized data envelopment analysis model. *European Journal of Operational Research, 95*(3), 491–510.